Solution Sheet 9

Exercise 1. (for credit, due on 23 November) Let X be a connected Riemann surface. We admit that X has a covering by a simply connected Riemann surface $\tilde{X}$. By the uniformization theorem, $\tilde{X}$ is biholomorphic to either $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$. We call X an annulus if $\pi_1(X, x_0) \cong \mathbb{Z}$.

(1) *(3 points)* Show that an annulus X is biholomorphic to either $\mathbb{D}^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$, $\mathbb{C}\backslash\{0\}$ or $C(a,b) = \{z \in \mathbb{C} \mid a < |z| < b\}$ for some $0 < a < b$.

(2) *(2 points)* Show that $C(a,b)$ is biholomorphic to $C(a',b')$ if and only if $\frac{a}{b} = \frac{a'}{b'}$.

Solution 1.

(1) Since $\tilde{X}$ is simply connected, the covering $\tilde{X} \to X$ is Galois and hence X is biholomorphic to $\tilde{X}/\Gamma$, where Γ is a discrete group acting freely and properly on $\tilde{X}$ through biholomorphisms. Furthermore, X is an annulus if and only if $\pi_1(X, x_0) \cong \Gamma \cong \mathbb{Z}$.

- Every non-trivial Möbius transformation has at least one fixed point on $\mathbb{P}^1$. This rules out the case $\tilde{X} = \mathbb{P}$.
- For $\tilde{X} = \mathbb{C}$, Γ must act by translations $z \mapsto az + b$. If $a \neq 1$, there is a fixed point $b/(1-a)$. Hence a fixed-point free $\mathbb{Z}$-action is generated by a translation $T : z \mapsto z + \omega$. We showed in Exercise sheet 1, Exercise 1 that $\mathbb{C}/\langle T \rangle$ is biholomorphic to $\mathbb{C}\backslash\{0\}$ by the map $z \mapsto e^{2\pi i z/\omega}$.
- Let us now consider the case $\tilde{X} = \mathbb{H}$. Automorphisms are Möbius transformations with real coefficients and positive determinant. Its fixed points satisfy

$$z = \frac{az+b}{cz+d} \quad \Longleftrightarrow \quad cz^2 + (d-a)z - b = 0.$$

  Since this is a quadratic equation with real coefficients, there are three possibilities:
  - Two complex conjugate roots, leading to one fixed point in $\mathbb{H}$ (elliptic)
  - Exactly one fixed point on $\mathbb{R} \cup \{\infty\}$ (parabolic),
  - Exactly two distinct fixed points on $\mathbb{R} \cup \{\infty\}$ (hyperbolic).

  A fixed-point free generator cannot be elliptic, so it is either:
  - Parabolic: The unique fixed point on $\mathbb{R} \cup \{\infty\}$ can be moved to ∞ via a Möbius transformation, after which the generator is conjugate to the translation $T : z \mapsto z + 1$. Then we get a biholomorphism $\mathbb{H}/\langle T \rangle \cong \mathbb{D}^*$.
  - Hyperbolic: The two distinct fixed points on $\mathbb{R} \cup \{\infty\}$ can be moved to 0 and ∞, after which the generator is conjugate to a dilation $A_\lambda : z \mapsto \lambda z$ with $\lambda > 1$. The logarithm map

$$w = \log z, \qquad 0 < \arg(z) < \pi$$

    induces a biholomorphism

$$\log : \mathbb{H} \to S := \{w \in \mathbb{C} : 0 < \Im w < \pi\},$$

    and satisfies

$$\log(\lambda z) = \log z + \log \lambda.$$

    Therefore

$$\mathbb{H}/\langle z \mapsto \lambda z \rangle \;\cong\; S/\langle w \mapsto w + \log \lambda \rangle.$$

    The map $\Phi(w) = e^{iw}$ gives a biholomorphism

$$\Phi : S \to C(e^{-\pi}, 1) = \{z \in \mathbb{C} : e^{-\pi} < |z| < 1\},$$

and furthermore $\Phi(w + \log \lambda) = \Phi(w)$. Therefore

$$\mathbb{H}/\langle z \mapsto \lambda z\rangle \cong C(e^{-\pi}, 1).$$

(2) Note that $C(a, b) \cong C(\frac{a}{b}, 1)$ by the map $f(z) = z/b$. We can represent

$$C(a, b) \cong C(\frac{a}{b}, 1) \cong \mathbb{H}/\langle z \mapsto \lambda z\rangle$$

with λ determined by a/b. Similarly $C(a', b') \cong C(\frac{a'}{b'}, 1) \cong \mathbb{H}/\langle z \mapsto \lambda' z\rangle$. Because $\mathbb{H}$ is simply connected, any biholomorphism $g : C(\frac{a}{b}, 1) \to C(\frac{a'}{b'}, 1)$ lifts to a biholomorphism $f : \mathbb{H} \to \mathbb{H}$ such that the diagram commutes:

$$\begin{tikzcd}
\mathbb{H} \arrow[r, "f"] \arrow[d, "p"'] & \mathbb{H} \arrow[d, "p'"] \\
C\left(\frac{a}{b}, 1\right) \arrow[r, "g"] & C\left(\frac{a'}{b'}, 1\right)
\end{tikzcd}$$

The commutative diagram implies that $p'(f(\lambda z)) = p'(f(z))$. Consequently $f(\lambda z)$ and $f(z)$ lie in the same fiber of p' and so they differ by an element of the deck transformation group of p' which is $\langle z \mapsto \lambda' z\rangle$. Hence there exists $n \in \mathbb{Z}$ such that $f(\lambda z) = (\lambda')^n f(z)$. Using that f has the form $z \mapsto \frac{az+b}{cz+d}$ with $ad - bc > 0$, one can check that the functional equation above forces $n = 1$ and $\lambda' = \lambda$. Consequently $\frac{a}{b} = \frac{a'}{b'}$.

Exercise 2. Let $\Lambda \subset \mathbb{C}$ be a rank-2 lattice. The Weierstrass $\wp$-function is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

We also define the Eisenstein series

$$g_2 = 60 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^4}, \quad g_3 = 140 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^6}.$$

(1) Show that the series defining $\wp$ converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \Lambda$.

(2) Show that $\wp$ is even, Λ-periodic and has double poles at Λ and no other poles. Show that $\wp'$ is odd, Λ-periodic and has triple poles at Λ and no other poles.

(3) Show that $\wp$ satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

(4) Let $\Delta := g_2^3 - 27g_3^2$ be the modular discriminant. Show that $4x^3 - g_2x - g_3$ has three distinct roots if and only if $\Delta \neq 0$.

Consider the elliptic curve $E \subset \mathbb{P}^2$ defined by

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3.$$

(5) Check that E is smooth if and only if $\Delta \neq 0$ and that the unique point with $Z = 0$ is $\mathcal{O} = [0 : 1 : 0]$.

(6) Show that the map $\phi : \mathbb{C}/\Lambda \to \mathbb{P}^2$ defined by

$$\phi([z]) = \begin{cases} [\wp(z) : \wp'(z) : 1], & z \notin \Lambda, \\ \mathcal{O}, & z \in \Lambda. \end{cases}$$

is holomorphic on all of $\mathbb{C}/\Lambda$.

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(7) Show that ϕ is bijective and conclude that $\phi : \mathbb{C}/\Lambda \to E$ is a biholomorphism. In particular, every complex torus of dimension 1 is algebraic.

Solution 2. Any standard textbook on modular forms or elliptic curves.

Exercise 3. (*) The Lagrangian of a simple pendulum is given by

$$\mathcal{L} = \frac{(\theta')^2}{2} - (1 - \cos\theta),$$

where θ is the angle from the vertical to the pendulum. Make precise the statement: The pendulum's motion is described by a closed periodic orbit on a suitable elliptic curve.